$$\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \boxed{\frac{\pi}{8}\log 2}$$

✎解説 tan 置換によって分母を払った後は ①log の中身を丁寧に分解していく方法と、②King Property を用いて tan の加法定理から求める方法の 2 パターンがある。

$x=\tan\theta$ とおくと、

$$\begin{aligned}\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx &= \int_0^{\frac{\pi}{4}} \frac{\log(1+\tan\theta)}{1+\tan^2\theta}(\tan\theta)'\,d\theta\\ &= \int_0^{\frac{\pi}{4}} \frac{\log(1+\tan\theta)}{1+\tan^2\theta}\cdot\frac{1}{\cos^2\theta}\,d\theta\\ &= \int_0^{\frac{\pi}{4}} \frac{\log(1+\tan\theta)}{\cos^2\theta+\tan^2\theta\cos^2\theta}\,d\theta\\ &= \int_0^{\frac{\pi}{4}} \frac{\log(1+\tan\theta)}{\cos^2\theta+\sin^2\theta}\,d\theta\\ &= \int_0^{\frac{\pi}{4}} \log(1+\tan\theta)\,d\theta(=\mathrm{I}_1)\end{aligned}$$

①log の中身を丁寧に分解していく方法

$$\begin{aligned}\mathrm{I}_1 &= \int_0^{\frac{\pi}{4}} \log\left(1+\frac{\sin\theta}{\cos\theta}\right)\,d\theta\\ &= \int_0^{\frac{\pi}{4}} \log\left(\frac{\cos\theta+\sin\theta}{\cos\theta}\right)\,d\theta\\ &= \int_0^{\frac{\pi}{4}} \log\left(\frac{\sqrt{2}\cos\left(\theta-\frac{\pi}{4}\right)}{\cos\theta}\right)\,d\theta\\ &= \int_0^{\frac{\pi}{4}} \log\sqrt{2}\,d\theta + \int_0^{\frac{\pi}{4}} \log\left\{\cos\left(\theta-\frac{\pi}{4}\right)\right\}\,d\theta - \int_0^{\frac{\pi}{4}} \log(\cos\theta)\,d\theta\\ &= \int_0^{\frac{\pi}{4}} \log\sqrt{2}\,d\theta + \int_{-\frac{\pi}{4}}^{0} \log(\cos\theta)\,d\theta - \int_0^{\frac{\pi}{4}} \log(\cos\theta)\,d\theta\\ &= \int_0^{\frac{\pi}{4}} \log\sqrt{2}\,d\theta + \int_{-\frac{\pi}{4}}^{0} \log(\cos\theta)\,d\theta + \int_0^{\frac{\pi}{4}} \log(\cos\theta)\,d\theta\\ &\qquad -2\int_0^{\frac{\pi}{4}} \log(\cos\theta)\,d\theta\\ &= \int_0^{\frac{\pi}{4}} \log\sqrt{2}\,d\theta + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log(\cos\theta)\,d\theta - 2\int_0^{\frac{\pi}{4}} \log(\cos\theta)\,d\theta\\ &= \int_0^{\frac{\pi}{4}} \log\sqrt{2}\,d\theta + 2\int_0^{\frac{\pi}{4}} \log(\cos\theta)\,d\theta - 2\int_0^{\frac{\pi}{4}} \log(\cos\theta)\,d\theta\\ &= \int_0^{\frac{\pi}{4}} \log\sqrt{2}\,d\theta = \Big[\theta\log\sqrt{2}\Big]_0^{\frac{\pi}{4}} = \frac{\pi}{4}\log\sqrt{2} = \frac{\pi}{8}\log 2\,_{//}\end{aligned}$$

②King Property を用いて tan の加法定理から求める方法
(答案では必ず $x=a+b-t$ の置換を明記すること)

I_1 において、$\theta = 0+\frac{\pi}{4}-t$ と置換すると

$$\begin{aligned}\mathrm{I}_1 &= \int_{\frac{\pi}{4}}^{0} \log\left\{\tan\left(0+\frac{\pi}{4}-t\right)+1\right\}\left(0+\frac{\pi}{4}-t\right)'\,dt\\ &= \int_{\frac{\pi}{4}}^{0} \log\left\{\tan\left(\frac{\pi}{4}-t\right)+1\right\}(-1)\,dt\\ &= \int_0^{\frac{\pi}{4}} \log\left\{\tan\left(\frac{\pi}{4}-t\right)+1\right\}\,dt\\ &= \int_0^{\frac{\pi}{4}} \log\left(\frac{\tan\frac{\pi}{4}-\tan t}{1+\tan\frac{\pi}{4}\cdot\tan t}+1\right)\,dt\ (\because \text{加法定理})\\ &= \int_0^{\frac{\pi}{4}} \log\left(\frac{\tan\frac{\pi}{4}-\tan t+1+\tan\frac{\pi}{4}\cdot\tan t}{1+\tan\frac{\pi}{4}\cdot\tan t}\right)\,dt\\ &= \int_0^{\frac{\pi}{4}} \log\left(\frac{1-\tan t+1+1\cdot\tan t}{1+1\cdot\tan t}\right)\,dt\\ &= \int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1+\tan t}\right)\,dt\\ &= \int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1+\tan\theta}\right)\,d\theta(=\mathrm{I}_2)\end{aligned}$$

ここで I_1 と I_2 を足し合わせて

$$\begin{aligned}2\cdot\mathrm{I}_1 &= \mathrm{I}_1+\mathrm{I}_2\\ &= \int_0^{\frac{\pi}{4}} \log(\tan\theta+1)\,d\theta + \int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1+\tan\theta}\right)\,d\theta\\ &= \int_0^{\frac{\pi}{4}} \log(\tan\theta+1)+\log\left(\frac{2}{1+\tan\theta}\right)\,d\theta\\ &= \int_0^{\frac{\pi}{4}} \log\left\{(\tan\theta+1)\cdot\frac{2}{1+\tan\theta}\right\}\,d\theta\\ &= \int_0^{\frac{\pi}{4}} \log 2\,d\theta = \Big[\theta\log 2\Big]_0^{\frac{\pi}{4}} = \frac{\pi}{4}\log 2\end{aligned}$$

$$\therefore \mathrm{I}_1 = \frac{\pi}{8}\log 2\,_{//}$$